Reconciling Stochastic Dominance and Indexation Approaches in Measuring Inequality

Michael Hoy*

University of Guelph

50 Stone Road East, Guelph, Ontario, N1G 2W1, Canada

Yu-Chin Hsu†

Academia Sinica

128 Academia Road, Section 2, Nankang, Taipei 11529, Taiwan

Rachel J. Huang‡

National Central University

300, Zhongda Rd., Zhongli District, Taoyuan City 32001, Taiwan

Larry Y. Tzeng§

National Taiwan University

300, Zhongda Rd., Zhongli District, Taoyuan City 32001, Taiwan

Lin Zhao¶

Southwestern University of Finance and Economics

555, Liutai Avenue, Wenjiang District, Chengdu, Sichuan 611130, China

April 6, 2021

*E-mail address: mhoy@uoguelph.ca.

†E-mail address: ychsu.econ@gmail.com.

‡E-mail address: rachel@ncu.edu.tw.

§E-mail address: tzeng@ntu.edu.tw.

¶E-mail address: zhaoliniss@swufe.edu.cn.

1 Introduction

There are two distinct approaches, developed in largely disconnected streams of research, to compare inequality between income distributions. One is the use of specific inequality indices (e.g., the Gini index) or indices drawn from a family of functions (e.g., the entropy measures). The other approach is the use of principles of stochastic dominance which has focused primarily on second and third degree stochastic dominance (SSD, TSD respectively). Adopting a specific inequality index has the advantage of leading to a complete ranking. However, the rankings from different inequality indices often disagree and it is not always clear what are the underlying normative principles responsible for the differences. The use of stochastic dominance to compare inequality of income distributions has the advantage that if one distribution is dominated by another according to one of SSD or TSD, then we can draw a firm conclusion about which distribution is more equal according to well understood and, for the most part, broadly acceptable normative principles. The main disadvantage, however, is that this method leads to only partial rankings. In this paper we develop a new approach which addresses these limitations.

The above noted limitations of existing methods for comparing inequality are clearly demonstrated in classic papers by Atkinson (1970, 2008). In these two papers, he uses data collected by Kuznets (1963) for 12 countries leading to 66 pairwise comparisons. He finds that SSD provides a ranking in only 16 cases (i.e., where Lorenz curves do not intersect). There are a further set of 47 comparisons with a single intersection of Lorenz curves. In these cases TSD delivered a ranking in only 13 cases, leaving 34 unresolved comparisons. A further 3 pairs had multiple intersecting Lorenz curves. Matters are not clearly resolved by using inequality indices. Using the so-called Atkinson family of inequality measures, he finds that rankings between pairs of income distributions are generally very sensitive to the choice of inequality aversion parameter. To the best of our knowledge, these limitations have not been successfully dealt with.

We develop a refinement of the stochastic dominance methodology which allows us to address the principle weaknesses of both the indexation and the existing stochastic dominance approaches. On the one hand, we are able to create a partial ranking finer than SSD and TSD, while on the other hand clearly demarcate which subsets of inequality indices from broad families would rank one distribution as more or less equal than another. In doing so, our approach illuminates the normative principles underlying specific iconic families of inequality indices in a systematic manner and in a way that is parallel to the creation of more finely grained rankings than is generated through existing SSD and TSD methods. It is in this sense that we reconcile the two approaches of indexation and stochastic dominance.

We call our approach θ-TSD$_h$ where θ is a parameter that represents the extent to which one may weaken or must strengthen the principle of *aversion to downside inequality* (ADI) in order to rank two distributions on the basis of inequality. h is a function to control for the property of the principle. In cases where the normative property ADI is stronger than required to rank a pair of distributions, we show the (maximal) extent to which one may weaken this requirement yet still obtain an unambiguous comparison. This implies a broadening of support for the ranking. In cases where the normative property ADI is not strong enough to rank a pair of distributions, we show the minimal extent to which one must strengthen this requirement in order to obtain an unambiguous comparison. In this latter case, we are able to determine in a precise manner the extent to which one must be more averse to downside inequality than is implied by the use of TSD in order to make a ranking. Although this implies a narrowing of support for the ranking, we determine the minimal extent to which one must do this to achieve a ranking of $\mathbf{y} \succeq \mathbf{x}$. Which direction of broadening or narrowing support relative to ADI in order to make a particular comparison is achieved by appropriate selection of the function h. We make explicit the normative properties inherent in the selection of h.

2 Examples and Preliminaries

To understand better how we address the limitations of both the stochastic dominance and indexation approaches, we introduce here the key features of our method and demonstrate our main contributions with the use of a simple example.

We begin with income distribution $\mathbf{y} = (5, 20, 30, 35)$ and consider the implications of a progressive transfer of amount $\Delta > 0$ from the second poorest to the poorest individual paired with a regressive transfer of amount $\delta > 0$ from the second richest to the richest individual. This generates distribution $\mathbf{y}^r = (5 + \Delta, 20 - \Delta, 30 - \delta, 35 + \delta)$. We restrict the sizes of Δ and δ so that the poorest individual in distribution $\mathbf{y}^r$ has higher income than the poorest individual in $\mathbf{y}$. The Rawlsian (lexicographic maximin) criterion would always judge such a composite transfer as welfare improving and so we follow Davies and Hoy (1994) in referring to such a pair of transfers as a Rawlsian Composite Transfer (RCT). Assume social planners judge welfare (or inequality) according to Utilitarian Social Welfare $U(\mathbf{y}) = \sum_{i=1}^{n} u(y_i)$ with social evaluation function $u(\cdot)$ satisfying $u' > 0$, $u'' < 0$. As is well known, all such social planners will assign less inequality or higher welfare to distribution $\mathbf{y}^r$ than $\mathbf{y}$, which we denote by $\mathbf{y}^r \succeq \mathbf{y}$, provided $\mathbf{y}^r$ dominates $\mathbf{y}$ by SSD. In our example, this requires that $\delta = 0$ (i.e., no regressive transfer is allowed at higher incomes is a necessary condition for all such social planners to agree).

Adding the normative requirement that $u''' > 0$ describes a set of social planners who subscribe to the normative criterion inherent in ADI. More specifically, choosing $\Delta = 5$, these social planners would conclude $\mathbf{y}^r \succeq \mathbf{y}$ provided $\delta \leq 5$.[1] If $\delta < 5$ ($\delta > 5$), however, it follows that the criterion of ADI is stronger (not strong enough) than required to conclude that $\mathbf{y}^r \succeq \mathbf{y}$. We refine TSD by replacing the condition $u''' > 0$ with $-u'''(y)/u''(y) \geq \theta/h(y)$, where θ is a constant and $h(y)$ could be either 1 or a positive affine function of y. We develop the analogous stochastic dominance condition which we refer to as θ-TSD$_h$. For $\theta > 0$, $u''' > 0$ is a necessary but not sufficient condition for $-u'''/u'' \geq \theta/h(y)$ while if $\theta < 0$, $u''' < 0$ is a necessary, but not sufficient, condition for $-u'''/u'' \geq \theta/h(y)$. Therefore, $\theta > 0$ ($\theta < 0$) implies a strengthening (weakening) of the criterion ADI.

An important advantage of our approach is that there is a parallel analysis to θ-TSD$_h$ which, under certain conditions, allows us to resolve the question of which inequality indices within certain iconic families rank $\mathbf{y}^r \succeq \mathbf{y}$ and which rank $\mathbf{y}^r \preceq \mathbf{y}$. We carry out this analysis by demonstrating that certain normatively interesting invariance conditions are equivalent to specific choices of function $h(y)$. For example, requiring that (relative) inequality be invariant to scaling (i.e., $\mathbf{y} \to \lambda\mathbf{y}$, where $\lambda > 0$), the corresponding choice of h is $h(y) = 1/y$. Suppose the Lorenz curve of $\mathbf{y}^r$ intersects that for $\mathbf{y}$ once from above. It follows that for each of our invariance classes,[2] there is a critical value of θ, call it θ^*, which allows us to

[1] Our example is constructed so that when $\Delta = \delta = 5$, the composite transfer has symmetric gap increasing and gap reducing effects between the first two and latter two pairs of individuals in the distribution y. The result is that the coefficient of variation is the same for the two distributions and so y^r "just" dominates y by TSD.

[2] Our invariance classes include translation invariance, $y \to y + \delta$, $\delta > 0$, embodied by Kolm's inequality

demarcate those inequality indices within the appropriate family that will conclude $\mathbf{y}^r \preceq \mathbf{y}$ from those that will conclude $\mathbf{y}^r \succeq \mathbf{y}$

We demonstrate with our example. Consider how the so-called entropy family of inequality measures assesses a degree of inequality and so a ranking for any pair of income distributions. The formula for this family is

$$I_{\alpha}^{\text{entropy}}(\mathbf{y}) = \frac{1}{\alpha(\alpha-1)} \left(E\left[\left(\frac{\mathbf{y}}{E[\mathbf{y}]} \right)^{\alpha} \right] - 1 \right), \ \alpha \neq 0, \ \alpha \neq 1. \tag{1}$$

where α is generally referred to as the inequality aversion index for this family. For values of $\alpha < 2$, this index is consistent with the normative requirement of ADI, albeit displaying stronger aversion to downside inequality as required for satisfaction of TSD. For values $\alpha > 2$, the index is not consistent with ADI and so places more importance on inequality the higher in the income distribution that it occurs. For $\alpha = 2$, the normative content of the index is on the border between agreeing or disagreeing with ADI (i.e., it "just" agrees with ADI). The relation between the parameter α and θ under θ-TSD$_h$ for scaling invariance is $\alpha = 2 - \theta$. It follows for the case of $\delta = 5$ that all indices within this family for which $\alpha < 2$ would conclude $\mathbf{y}^r \succ \mathbf{y}$, while those for which $\alpha > 2$ would conclude $\mathbf{y} \succ \mathbf{y}^r$, and in the case of $\alpha = 2$ we have $\mathbf{y}^r \approx \mathbf{y}$. Hence, in this case the critical value of θ is $\theta^* = 0$. If $\delta = 2$, the requirement of TSD is stronger than needed to conclude that $\mathbf{y}^r \succeq \mathbf{y}$. This suggests there is a broader range of indices within the entropy family that would conclude $\mathbf{y}^r \succeq \mathbf{y}$. In this case it turns out that the critical value of θ is -1.66 ($\alpha = 3.66$) and so a larger set of indices ($\alpha < 3.66$) would conclude $\mathbf{y} \succeq \mathbf{y}^r$. On the other hand, if $\delta > 5$, a higher degree of aversion to downside inequality than reflected by TSD (or $\alpha = 2$ for the entropy family) would have to hold for a social planner to conclude $\mathbf{y}^r \succeq \mathbf{y}$. For example, if $\delta = 7$, the critical value of θ is 0.62 ($\alpha = 1.38$) and so a smaller set of indices ($\alpha < 1.38$) would conclude would conclude $\mathbf{y} \succeq \mathbf{y}^r$.

index (1976) as well as a mixture of scale and translation invariance, which corresponds to the normative foundation of intermediate inequality indices due to Pfingsten (1986) and Bossert and Pfingsten (1990).

3 Conclusion

In this paper, we consolidate two distinct approaches, i.e., the use of specific inequality indices and the use of principles of stochastic dominance, to compare downside inequality between income distributions. Specifically, for the Utilitarian Social Welfare function, $U(\mathbf{y}) = \sum_{i=1}^{n} u(y_i)$ with $u' > 0$, $u'' < 0$, we add in a lower bound of $-u'''(y)/u''(y)$ to control for the degree of downside inequality aversion. We first provide the economic intuition of the constraint, i.e., $-u'''(y)/u''(y) \geq \theta/h(y)$, where θ is a parameter, and $h(y) > 0$. It could be viewed as that there is a benchmark social planner whose $-u'''(y)/u''(y)$ is equal to $\theta/h(y)$.

We then provide the dominance condition such that all inequality-averse social planner whose $-u'''(y)/u''(y) \geq \theta/h(y)$ would agree that one income distribution is preferred to another one. The dominance is referred to as θ-TSD$_h$. The function $h(y)$ can controls for the property of the inequality. We show that the ranking of inequality is invariant to translation when $h(y)$ is equal to one. When the invariance of scale is required, we can choose $h(y) = 1/y$. We further show that by choosing $h(y) = 1/\left[\mu y + (1-\mu)\right]$, the ranking is invariant of μ-transform. The parameter θ can control for the intensity of downside inequality aversion under different types of h. When $\theta = 0$, our ranking is reduced to TSD. It is because the condition $-u'''(y)/u''(y) \geq 0$ is equivalent to require $u''' > 0$. When $\theta < 0$, it means that some social planner could exhibit $u''' < 0$ for some y. When $\theta > 0$, it means that all social planner should exhibit $u''' > 0$ and the degree of downside inequality aversion is large.

Furthermore, for a given $h(y)$, we show to compare any two income distributions, there is a critical value of θ, referred as θ^*, such that all social planners with parameter $\theta < \theta^*$ would prefer one distribution and all others with parameter $\theta > \theta^*$ would prefer the other distribution. This critical parameter itself could serve as an index to evaluate downside inequality. Moreover, this critical parameter gives economic foundations to the parameters contained in the existing inequality measures. For example, by choosing $h(y) = 1/y$, the critical θ^* is monotonic to the parameter α in the entropy family of inequality measures as shown in Equation (1). We find that $\alpha = 2 - \theta^*$ when the Lorenz curves intersect each other once. By choosing $h(y) = 1$, the critical θ^* for two income distributions whose Lorenz curves intersect once is equal to the parameter α in the Kolm family of inequality indices:

$$K_\alpha(F) = \frac{1}{\alpha} \ln E_F \left[e^{\alpha(\mu - x)} \right].$$

In summary, by adding a lower bound on the degree of downside inequality aversion, we establish partial ranking criteria for downside inequality aversion. From these partial ranking criteria, we further construct an index to evaluate downside inequality. We demonstrate that our index gives economic foundations for parameters of the existing inequality measures. Furthermore, we will use empirical data to demonstrate how to apply our orders.

References

[1] Atkinson, A.B., 1970. On the measurement of inequality. J. Econ. Theory. 2, 244–263.

[2] Atkinson, A.B., 2008. More on the measurement of inequality. J. Econ. Inequality. 6, 277–283.

[3] Bossert, W., Pfingsten, A., 1990. Intermediate inequality: concepts, indices, and welfare implications. Mathematical Social Sciences. 19, 117–134.

[4] Davies, J.B., Hoy, M., 1994. The normative significance of using third-degree stochastic dominance in compariing income distributions. J Econ Theory. 64, 520–530.

[5] Kolm, S., 1976. Unequal inequalities: I. J Econ Theory. 12, 416–442.

[6] Kuznets, S., 1963. Quantitative aspects of economic growth of nations. Econ Dev Cult Change 11, 1–80.

[7] Pfingsten, A., 1986. Distributionally-neutral tax changes for different inequality concepts. J. Pub. Econ. 30, 385–393.